TEACHERS FORUM®

QUESTION BANK

(solved)

Based on CBSE Previous years' question papers

Class XII

MATHEMATICS

SUBJECT EXPERTS

CONTENTS

1 RELATIONS AND FUNCTIONS

1 MARKS

1. A relation in a set A is called __________ relation, if each element of A is related to itself. **(2020)**

Ans. reflexive

2. The least value of the function $f(x) = ax + \frac{b}{x}$ (a > 0, b > 0, x > 0) is ______. **(2020)**

Ans. Here $f(x) = ax + \frac{b}{x}$; a > 0, b > 0, $x > 0$

$\therefore \quad f'(x) = a - \frac{b}{x^2}$ and $f''(x) = \frac{2b}{x^3}$

For critical points, $f'(x) = a - \frac{b}{x^2} = 0 \Rightarrow x = \sqrt{\frac{b}{a}}$

As $f''(x)\left(\sqrt{\frac{b}{a}}\right) = \frac{2b}{\frac{b^{3/2}}{a^{3/2}}} = \frac{2a^{3/2}}{\sqrt{b}} > 0$ so, $f(x)$ has least value at $x = \sqrt{\frac{b}{a}}$

Also, the least value of functions $f\left(\sqrt{\frac{b}{a}}\right) = a\sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$

3. If f : R $\to$ R is given by $f(x) = (3 - x^3)^{1/3}$, find fof (x). **(2019)**

Ans. fof(x) = f(f(x)) = f$((3 - x^3)^{1/3})$

$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3} = x$

4. If a ∗ b denotes the larger of 'a' and 'b' and if a o b = (a ∗ b) + 3, then write the value of (5) o (10), where ∗ and o are binary operations. **(2018)**

Ans. 5 o 10 = (5 ∗ 10) + 3 = 10 + 3 = 13

5. Find the identity element in the set Q^+ of all positive rational numbers for the operation ∗ defined by $a * b = \frac{3ab}{2}$ for all a, b $\in Q_+$. **(2018)**

Ans. $\frac{3ae}{2} = a, \quad e = \frac{2}{3}$

6. Let ∗ be a 'binary' operation on N given by a ∗ b = LCM (a, b) for all a, b $\in$ N. Find 5 ∗ 7. **(2012)**

Ans. 35

7. The binary operation ∗ : R x R $\to$ R is defined as a ∗ b = 2a + b. Find (2 ∗ 3) ∗ 4.

Ans. 18 **(2012)**

8. State the reason for the relation R in the set {1, 2, 3} given by R= {(l, 2), (2, 1)} not to be transitive. **(2011)**

Ans. $(1, 2) \in R, (2, 1) \in R$ but $(1, 1) \notin R$

9. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. State whether f is one-one or not. **(2011)**

Ans. f is a one-one function

10. If the binary operation * on the set of integers Z, is defined by a * b = a + 3b^2, then find the value of 2 * 4. **(2009)**

Ans. 50

11. Let* be a binary operation on N given by a * b = HCF (a, b), a, b $\in$ N.

Write the value of 22*4. **(2009)**

Ans. 2

12. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, find (fog) (7) **(2008)**

Ans. 7

13. Let * be a binary operation defined by a * b = 2a + b – 3. Find 3 * 4. **(2008)**

Ans. 3* 4 = 2 × 3 + 4 − 3 = 7

2 MARKS

14. If $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$, then show that $(fof)(x) = x$, for all $x \neq \frac{2}{3}$. Also write inverse of f.

Ans. Given $f(x) = \frac{4x+3}{6x-4}\ x \neq \frac{2}{3},$ **(2020)**

$$fof(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right) = \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x.$$

Let $y = f(x) = \frac{4x+3}{6x-4}$

$\Rightarrow\ 6xy - 4y = 4x + 3$

$\Rightarrow\ 6xy - 4x = 3 + 4y$

$\Rightarrow\ x = \frac{3+4y}{6y-4}$ i.e., $f^{-1}(x) = \frac{3+4x}{6x-4}, x \neq \frac{2}{3}$

15. Check if the relation R in the set R of real numbers defined as R = {(a, b) : a < b} is

(i) symmetric, (ii) transitive **(2020)**

Ans. We have $R = \{(a, b) : a < b\}$ where $a, b \in R$.

(i) Symmetry : Observe that 1 < 2 is true but 2 < 1 is not true.

That is, $(1,2) \notin R$ but $(2,1) \in R$ so, R is not symmetric.

(ii) Transitivity : Observe that if a < b and b < c are both true then, a < c is also true, for all real numbers a, b, c.

That is, $(a, b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ So, R is transitive.

16. Let A = N X N be the set of all ordered pairs of natural numbers and R be the relation on the set A defined by (a, b) R (c, d) if ad = bc. Show that R is an equivalence relation.

Ans. For reflexive : **(2019)**

As $ab = ba$

$\Rightarrow (a, b) R (a, b) \therefore$ R is reflexive

For symmetric:

Let (a, b,) R (c, d)

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c, d) R (a, b) \therefore$ R is symmetric

For transitive:

Let $a, b, c, d, e, f \in N$

Let (a, b) R(c, d) and (c, d) R(e, f)

$\Rightarrow ad = bc$ and $cf = de$

$\Rightarrow d = \frac{cf}{e}$

$\therefore a\left(\frac{cf}{e}\right) = bc$

$acf = bce \Rightarrow af = be$

$\Rightarrow (a, b) R (e, f) \therefore$ R is transitive

Since R is reflexive, symmetric and transitive $\therefore$ R is an equivalence relation.

17. Show that $f : R - \{2\} \rightarrow R - \{1\}$ defined by $f(x) = \frac{x}{x-2}$ is one-one. Also, if $g : R - \{1\} \rightarrow R - \{2\}$ is defined as $g(x) = \frac{2x}{x-1}$, find gof (x). **(2019)**

Ans. Let $x_1, x_2 \in R - \{2\}$

Let $f(x_1) = f(x_2)$

$$\frac{x_1}{x_1 - 2} = \frac{x_2}{x_2 - 2} \Rightarrow x_1(x_2 - 2) = x_2(x_1 - 2)$$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ f is one-one.

Now, $gof(x) = g(f(x)), x \in R - \{2\}$

$$= g\left(\frac{x}{x-2}\right) = \frac{2\left(\frac{x}{x-2}\right)}{\frac{x}{x-2} - 1} = x$$

18. Prove that the relation R in R, the set of all real numbers, defined as R = {(a, b) : a ≤ b²} is neither reflexive nor transitive. **(2019)**

Ans. $0.1 \in R$ with $0.1 \nleq (0.1)^2$ $\quad \therefore (0.1, 0.1) \notin R$

$\therefore$ R is not reflexive

$10, 4, 3 \in R$ with $10 \leq 4^2, 4 \leq 3^2$ but $10 \nleq 3^2$

$\therefore (10, 4), (4, 3) \in R$ but $(10, 3) \in R \therefore$ R is not transitive

19. Consider f : + $R_+ \rightarrow [4, \infty]$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse of f given by $f^{-1}(y) = \sqrt{y - 4}$, where R_+ is the set of all non-negative real numbers. **(2019)**

Ans. Let $x_1, x_2 \in R_+$

Such that $f(x_1) = f(x_2) \Rightarrow x_1^2 + 4 = x_2^2 + 4 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

$\therefore f$ is a one-one function.

Now, $x^2 \geq 0 \Rightarrow x^2 + 4 \geq 4 \Rightarrow f(x) \in [4, \infty)$

$\Rightarrow R_f = [4, \infty)$ = Co-domain $(f) \therefore f$ is an onto function.

$\therefore f$ is an invertible function.

$y = 2 f(x) = x + 4 \Rightarrow x = \sqrt{y - 4} = f^{-1}(y) = \sqrt{y - 4}$

20. Prove that if E and F are independent events, then the events E and F' are also independent. **(2017)**

Ans.

$$P(E \cap F') = P(E) - P(E \cap F)$$
$$= P(E) - P(E) . P(F)$$
$$= P(E)[1 - P(F)]$$
$$= P(E)P(F')$$

$\Rightarrow$ E and F' are independent events.

21. Show that the relation R defined by (a, b) R (c, d) $\Rightarrow$ a + d = b + c on the set N × N is an equivalence relation. **(2008)**

Ans. (i) (a, b) R (c, d) $\Rightarrow$ a+d = b+c

where (a, b), (c, d) $\in$ N x N

(a, b) R (a, b) $\Rightarrow$ a + b = b + a $\Rightarrow$ True

R is Reflexive

(ii) (a, b) R (c, d) $\Rightarrow$ a + d = b + c $\Rightarrow$ b + c = a + d

= c + b = d + a

$\Rightarrow$ (c, d) R (a, b)

Hence R is Symmetric

(iii) Let (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ a + d = b + c and c + f = d + e

Adding we get

a + d + c + f = b + c + d + e

$\Rightarrow$ a + f = b + e $\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation

3 MARKS

22. Let R be the relation defined in the set A = {1, 2, 3, 4, 5, 6, 7, 8, 9} by R = {(x, y) : x, y $\in$ A, x and y are either both odd or both even}. Show that R is an equivalence relation. Write all the equivalence classes of set A. **(2018)**

Ans. Reflexive : Clearly (x, x) $\in$ R $\because$ x is either odd or even. So R is reflexive

Symmetric

Let (x, y) $\in$ R

$\therefore$ x, y are both either odd or even

$\Rightarrow$ y, x are both either odd or even

$\Rightarrow$ (y, x) $\in$ R, So R is symmetric

Transitive

Let (x, y) $\in$ R and (y, z) $\in$ R

Case (i) x and y are both odd so y and z are both odd

$\therefore$ x and z are both odd

$\therefore$ (x, z) $\in$ R

Case (ii) x and y are both even, so y and z are both even

$\therefore$ x and z are both even

$\therefore$ (x, z) $\in$ R

Thus (x, y) and $(y, z) \in R \Rightarrow (x, z) \in R$

R is transitive

As R is reflexive, symmetric and transitive so R is an equivalence relation.

[1] = {1, 3, 5, 7, 9}

[2] = {2, 4, 6, 8} are required equivalence classes

23. Let A be the set of all real numbers except -1 and let $*$ be a binary operation on A defined by $a * b = a + b + ab$, for $\forall$ a, b $\in$ A. Prove that (i) $*$ is commutative and associative, and (ii) number 0 is its identity element. **(2018)**

Ans. $a * b = a + b + ab$

$b * a = b + a + ba$

$a * b = b * a \ \forall\ a, b \in A$

So $*$ is commutative

$a * (b * c) = a * (b + c + bc) = a + b + c + bc + ab + ac + abc$

$(a * b) * c = (a + b + ab) * c = a + b + ab + c + ac + bc + abc$

$a * (b * c) = (a * b) * c \ \forall\ a, b, c \in A$

$\therefore$ $*$ is associative

$a * 0 = a + 0 + a \cdot 0 = a$

$0 * a = 0 + a + 0 \cdot a = a$

and $0 \in A \therefore 0$ is the identity element

24. Show that the relation R on the set Z of all integers defined by $(x, y) \in R \Leftrightarrow (x - y)$ is divisible by 3 is an equivalence relation. **(2018)**

Ans. $(x - x) = 0$ is divisible by 3 for all $x \in z$. So, $(x, x) \in R$

$\therefore$ R is reflexive.

$(x - y)$ is divisible by 3 $\Rightarrow (y - x)$ is divisible by 3.

So (x, y) Î R $\Rightarrow$ $(y, x) \in R$, x, y $\in$ z

$\Rightarrow$ R is symmetric.

$(x - y)$ is divisible by 3 and $(y - z)$ is divisble by 3.

So $(x - z) = (x - y) + (y - z)$ is divisible by 3.

Hence $(x, z) \in R \Rightarrow$ R is transitive

$\Rightarrow$ R is an equivalence relation

25. Show that zero is the identity for this operation $*$ and each element 'a' $\neq 0$ of the set is invertible with 6 – a, being the inverse of 'a'. **(2018)**

Ans.

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$a * 0 = a + 0 = a \ \forall\ a \in A \Rightarrow 0$ is the identifty for $*$.

Let $b = 6 - a$ for $a \neq 0$

Since $a + b = a + 6 - a < 6$

$\Rightarrow a * b = b * a = a + 6 - a - 6 = 0$

Hence $b = 6 - a$ is the inverse of a.

26. Consider f : **R**+ $\rightarrow [-9, \infty]$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{54 + 5y} - 3}{5}\right)$. **(2017)**

Ans. $f : R_+ \rightarrow [-9, \infty)$; $f(x) = 5x^2 + 6x - 9$; $f^{-1}(y) = \frac{\sqrt{54 + 5y} - 3}{5}$

$$fof^{-1}(y) = 5\left\{\frac{\sqrt{54 + 5y} - 3}{5}\right\}^2 + 6\left\{\frac{\sqrt{54 + 5y} - 3}{5}\right\} - 9 = y$$

$$f^{-1} of (x) = \frac{\sqrt{54 + 5(5x^2 + 6x - 9)} - 3}{5} = x$$

Hence 'f' is invertible with $f^{-1}(y) = \frac{\sqrt{54 + 5y} - 3}{5}$

27. A binary operation * is defined on the set $x = R - \{-1\}$ by

$$x * y = x + y + xy, \ \forall\ x, y \in X.$$

Check whether * is commutative and associative. Find its identity element and also find the inverse of each element of X. **(2017)**

Ans. (i) commutative : let $x, y \in R - \{-1\}$ then

$x * y = x + y + xy = y + x + yx = y * x$

$\therefore$ * is commutative

(ii) Associative : let $x, y, z \in R - \{-1\}$ then

$x * (y * z) = x * (y + z + yz) = x + (y + z + yz) + x(y + z + yz)$

$= x + y + z + xy + yz + zx + xyz$

$(x * y) * z = (x + y + xy) * z = (x + y + xy) + z + (x + y + xy) . z$

$= x + y + z + xy + yz + zx + xyz$

$x * (y * z) = (x * y) * z$

$\therefore$ * is Associative

(iii) Identity Element : let $e \in R - \{-1\}$ such that $a * e = e * a = a \ \forall \ a \in R - \{-1\}$

$\therefore a + e + ae = a$

$\Rightarrow \ e = 0$

(iv) Inverse : let $a * b = b * a = e = 0$; $a, b \in R - \{-1\}$

$\Rightarrow a + b + ab = 0$

$\therefore b = \frac{-a}{1+a}$ or $a^{-1} = \frac{-a}{1+a}$

28. Let $f : N \rightarrow N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \rightarrow S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$. **(2016)**

Ans. Let $x_1, x_2 \in N$ and $f(x_1) = f(x_2)$

$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow (x_1 - x_2)(9x_1 + 9x_2 + 6) = 0$

$\Rightarrow x_1 - x_2 = 0$

or $x_1 = x_2$ as $(9x_1 + 9x_2 + 6) \neq 0$, $x_1, x_2 \in N$

$\therefore$ f is a one-one function

$f : N \rightarrow S$ is ONTO as co-domain = Range

Hence f is invertible

$y = 9x^2 + 6x - 5 = (3x + 1)^2 - 6 \Rightarrow x = \frac{\sqrt{y+6} - 1}{3}$

$f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$, $y \in S$

$f^{-1}(43) = \frac{\sqrt{49} - 1}{3} = 2$

$f^{-1}(163) = \frac{\sqrt{169} - 1}{3} = 4$

29. If f, g : R $\rightarrow$ R be two functions defined as $f(x) = | x | + x$ and $g(x) = | x | - x$, $\forall \ x \in R$.

Then find fog and gof. Hence find fog(–3), fog(5) and gof (–2). **(2016)**

Ans. $f(x) = | x | + x$ and $g(x) = | x | - x$, $\forall \ x \in R$

$(fog)(x) = f(g(x))$

$= ||x| - x| + |x| - x$

$$(gof)(x) = g(f(x))$$
$$= ||x| + x| - |x| - x$$
$$(fog)(-3) = 6$$
$$(fog)(5) = 0$$
$$(gof)(-2) = 2$$

30. Determine whether the relation R defined on the set of all real R numbers as R = {(a, b) : a, b $\in$ R and a – b + $\sqrt{3}$ $\in$ S, where S is the set of all irrational numbers}, is reflexive, symmetric and transitive. **(2015)**

Ans. Here R = { (a, b) : a, b $\in \Re$ and a - b + $\sqrt{3}$ $\in$S, where

S is the set of all irrational numbers.}

(i) $\forall$ a $\in \Re$, (a, a) $\in$ R as a - a + $\sqrt{3}$ is irrational

$\therefore$ R is reflexive

(ii) Let for a, b $\in \Re$, (a, b) $\in$ R i. e., a - b + $\sqrt{3}$ is irrational

a - b + $\sqrt{3}$ is irrational $\Rightarrow$ b - a + $\sqrt{3}$ $\in$ S $\therefore$ (b, a) $\in$ R

Hence R is symmetric

(iii) Let (a, b) $\in$ R and (b, c) $\in$ R , for a, b, c $\in \Re$

a - b + $\sqrt{3}$ $\in$ S and b - c + $\sqrt{3}$ $\in$ S

adding to get a - c + 2 $\sqrt{3}$ $\in$ S Hence (a, c) $\in$ R

$\therefore$ R is Transitive

31. Let A = R x R and $*$ be the binary operation on A defined by (a, b) $*$ (c, d) = (a + c, b + d). Prove that $*$ is commutative and associative. Find the identity element for * on A. Also write the inverse element of the element (3, – 5) in A. **(2015)**

Ans. $\forall$ a, b, c, d, e, f $\in$ R

$$((a, b) * (c, d) * (e, f) = (a + c, b + d) * (e, f)$$
$$= (a + c + e, b + d + f) \quad \rightarrow(3)$$
$$(a, b) * ((c, d) * (e, f)) = (a, b) * (c + e, d + f)$$
$$= (a + c + e, b + d + f) \quad \rightarrow(4)$$

$*$ is Associative

Let (x, y) be on identity element in R x R

$$\Rightarrow (a, b) * (x, y) = (a, b) = (x, y) * (a, b)$$
$$\Rightarrow a + x = a, b + y = b$$
$$x = 0 , y = 0$$

$\therefore$ (0, 0) is identity element

Let the inverse element of (3, – 5) be (x_1, y_1)

$\Rightarrow$ $(3, -5) * (x_1, y_1) = (0, 0) = (x_1, y_1) * (3, -5)$

$3 + x_1 = 0, \ -5 + y_1 = 0$

$\Rightarrow$ $x_1 = -3, \ y_1 = 5$

$\Rightarrow$ (– 3, 5) is an inverse of (3, – 5)

32. If $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x-1}{x^2+1}$ and $h(x) = 2x - 3$, then find $f^{|}[\, h^{|}\{g^{|}(x)\}]$. **(2015)**

Ans. $f(x) = \sqrt{x^2+1}$ $g(x) = \dfrac{x-1}{x^2+1}$, $h(x) = 2x - 3$

Differentiating w.r.t. "x", we get

$$f^{|}(x) = \frac{x}{\sqrt{x^2+1}},\ g^{|}(x) = \frac{1-2x-x^2}{(x^2+1)^2},\ h^{|}(x) = 2$$

$$f^{|}(h^{|}(g^{|}(x))) = \frac{2}{\sqrt{5}}$$

33. Show that $f: N \rightarrow N$, given by

$f(x)\begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$ is both one-one and onto. **(2012)**

Ans. Let x_1 be odd and x_2 be even and suppose $f(x_1) = f(x_2)$

$\Rightarrow x_1 + 1 = x_2 - 1 \Rightarrow x_2 - x_1 = 2$ which is not possible

simlarly, if x_2 is odd and x_1 is even, not possible to have $f(x_1) = f(x_2)$

Let x_1 and x_2 be both odd $\Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

simlarly, if x_1 and x_2 are both even, then also $x_1 = x_2$

$\therefore f$ is one – one

Also, any odd number 2r + 1 in co-domain N is the image of (2r + 2) in domain N and any even number 2r in the co-domain N is the image of (2r – 1) in domain N

$\Rightarrow f$ is on to

34. Consider the binary operations $* : R \times R \rightarrow R$ and $o : R \times R \rightarrow R$ defined as $a * b = |a - b|$ and $a \text{ o } b = a$ for all $a, b \in R$. Show that ' * ' is commutative but not associative, 'o' is associative but not commutative. **(2012)**

Ans. $a * b = |a - b|$ and $b * a = |b - a|$. also $|a - b| = |b - a|$ for all $a, b \in R$

$\therefore a * b = b * a \Rightarrow *$ is commutative

Also, $[(-2) * 3] * 4 = |-2-3| * 4 = 5 * 4 = |5-4| = 1$

and $(-2) * [3 * 4] = (-2) * |3-4| = -2 * 1 = |-2-1| = 3$

$\therefore *$ is not associative

2o3 = 2 and 3o2 = 3 $\Rightarrow$ o is not commutative

for any a, b, c $\in$ R (a o b) oc = a o c = a and ao (boc) = a o b = a

$\Rightarrow$ o is a associative

35. Consider the binary operation $*$ on the set {1, 2, 3, 4, 5} defined by a $*$ b = min. {a, b}. Write the operation table of the operation $*$. **(2011)**

Ans.

a $*$ b	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

36. Let f :R $\rightarrow$R be defined as $f(x)$ = 10x + 7. Find the function g :R $\rightarrow$ R such that gof = fog = I_R. **(2011)**

Ans.

$$\text{Let } y = 10x + 7$$

$$\therefore x = \frac{1}{10}(y - 7)$$

$$\text{let } g(y) = \frac{1}{10}(y - 7)$$

$$\therefore \text{gof}(x) = g(10x + 7) = \frac{1}{10}(10x + 7 - 7) = x \Rightarrow I_R = \text{gof}$$

$$\text{and fog}(y) = f\left(\frac{1}{10}(y - 7)\right) = 10\left(\frac{1}{10}(y - 7)\right) + 7 = y \Rightarrow I_R = \text{fog}$$

$$\text{Hence } g(y) = \frac{1}{10}(y - 7)$$

37. A binary operation $*$ on the set {0, 1, 2, 3, 4, 5} is defined as : **(2011)**

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element 'a' of the set is, invertible with 6 – a, being the inverse of 'a'.

Ans. $\left.\begin{array}{l} \text{since } a * 0 = a + 0 = a \\ \text{and } 0 * a = 0 + a = a \end{array}\right\} \forall\, a \in \{0, 1, 2, 3, 4, 5\}$

$\therefore$ 0 is the identity for $*$.

Also, $\forall$ a $\in$ {0, 1, 2, 3, 4, 5}, a $*$ (6 – a) = a + (6 – a) – 6 = 0 (which is identity)

$\therefore$ Each element ‘a’ of the set is invertible with (6 – a), being the inverse of ‘a’.

38. Prove that the relation R in the set A = {1, 2, 3, 4,5} given by R = { (a, b) : | a - b | is even}, is an equivalence relation. **(2009)**

Ans. (i) for all a $\in$A, (a, a) $\in$R $\therefore$ |a – a| = o is even

$\therefore$ R is reflexive in A

(ii) for all a, b $\in$A, (a,b) $\in$R $\Rightarrow$ (b, a) $\in$R

$\therefore$ if | a – b | is even then | b – a | is also even $\Rightarrow$ R is symmetric in A.

(iii) for all a,b, c $\in$A

(a, b) $\in$R and (b, c) $\in$R then (a, c) $\in$R

$\therefore$ | a – b | is even, | b – c | is even, then | a – c | will also be even

Hence, R is an equivalence relation in A

4 MARKS

39. Let A = R – {1}. If f : A $\to$ A is a mapping defined by $f(x) = \frac{x-2}{x-1}$, show that f is bijective, find f^{-1}. Also find :

(i) x if $f^{-1}(x) = \frac{5}{6}$

(ii) f^{-1} (2) **(2017)**

Ans. f : A $\to$ A

Let $x_1, x_2 \in$ A such that $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1-2}{x_1-1} = \frac{x_2-2}{x_2-1}$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one

Now $y = \frac{x-2}{x-1} \Rightarrow x - 2 = xy - y$

$\Rightarrow x(y - 1) = y - 2$

$\Rightarrow x = \frac{y-2}{y-1}$

For each y $\in$ A = R – {1}, there exists $x \in$ A

Thus f is onto. Hence f is bijective

and $f^{-1}(x) = \frac{x-2}{x-1}$

(i) $f^{-1}(x) = \frac{5}{6} \Rightarrow \frac{x-2}{x-1} = \frac{5}{6} \Rightarrow x = 7$

(ii) $f^{-1}(2) = 0$

40. Let A = R – {2} and B = R – {1}. Consider the function f : A $\to$ B defined by $f(x) = \frac{x-1}{x-2}$

Prove that f is one-one and onto function. Also, find $f^{-1}(x)$. If $f^{-1}(x) = 7$, find x. **(2017)**

Ans. For one-one

Let x_1, x_2 A $\in$ such that $f(x_1) = f(x_2)$

$$\frac{x_1 - 1}{x_1 - 2} = \frac{x_2 - 1}{x_2 - 2}$$

$$\Rightarrow x_1 x_2 - 2x_1 - x_2 + 2 = x_1 x_2 - x_1 - 2x_2 + 2$$

$$\Rightarrow -x_1 = -x_2$$

$$\Rightarrow x_1 = x_2$$

So f is one-one

For onto :

Let y $\in$ B

$$\text{Let } f(x) = y$$

$$\text{i.e., } \frac{x - 1}{x - 2} = y$$

$$\Rightarrow x - 1 = xy - 2y$$

$$\Rightarrow x - xy = 1 - 2y$$

$$\Rightarrow x = \frac{1 - 2y}{1 - y} \in A \ \forall\ y \in B$$

$\therefore f$ is onto 2

So f is invertible & f^{-1}: B $\rightarrow$ A defined as $f^{-1}(x) = \dfrac{1 - 2x}{1 - x}$

$$\Rightarrow \frac{1 - 2x}{1 - x} = 7 \qquad \text{(since } f^{-1}(x) = 7)$$

$$\Rightarrow 1 - 2x = 7 - 7x$$

$$\Rightarrow 5x = 6$$

$$\Rightarrow x = \frac{6}{5}$$

41. Let * be a binary operation on R – {–1}, defined by a $*$ b = a + b + ab for all a, b $\in$ R – {–1}. Prove that $*$ is commutative and associative on R – {–1}. Find the identity element for $*$ on R – {–1}. **(2017)**

Ans. Let a, b $\in$ R – {–1}

$$a * b = a + b + ab$$

$$b * a = b + a + ba$$

$$\therefore a * b = b * a \ \forall\ a, b \in R \sim \{-1\}$$

$\therefore$ $*$ is commutative

Let a, b, c $\in$ R – {–1}

$$a * (b * c) = a * (b + c + bc)$$

$$= a + b + c + bc + a(b + c + bc)$$

$= a + b + c + bc + ab + ac + abc$

$(a * b) * c = (a + b + ab) * c$

$= a + b + ab + c + (a + b + ab)c$

$= a + b + ab + c + ac + bc + abc$

So $a * (b * c) = (a * b) * c$

$\therefore *$ is associative

For identity

Let $e \in R - \{-1\}$ be identity element

$\therefore a * e = a = e * a$

$a * e = a$

$\therefore a + e + ae = a$

$e(1 + a) = 0$

$\Rightarrow e = 0$ as $1 + a \neq 0$

$\therefore$ 0 is the identity element

42. Let A = R – {3}, B = R – {1}. Let f : A $\to$ B be defined by $f(x) = \frac{x - 2}{x - 3}, \forall x \in A$

Show that f is bijective. Also, find

(i) x, if $f^{-1}(x) = 4$

(ii) $f^{-1}(7)$ **(2017)**

Ans. Let $x_1, x_2 \in$ A and $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$

$$\Rightarrow x_1 = x_2$$

Hence f is a one-one function

$$\text{Let } y = \frac{x - 2}{x - 3} \text{ for } y \in R - \{1\}$$

$$\Rightarrow x = \frac{3y - 2}{y - 1}; \ y \neq 1$$

$\therefore \forall y \in R - \{1\}, x \in R - \{3\}$

i.e. Range of f = co-domain of f.

Hence f is onto and so bijective.

Also, $f^{-1}(x) = \frac{3x - 2}{x - 1}\ x \neq 1$

(i) Now, $f^{-1}(x) = 4$

$\Rightarrow \frac{3x - 2}{x - 1} = 4$

$\Rightarrow x = 2$

(ii) and $f^{-1}(7) = \frac{19}{6}$

43. Let A = R x R and let $*$ be a binary operation on A defined by (a, b) $*$ (c, d) = (ad + bc, bd) for all (a, b), (c, d) $\in$ R x R.

(i) Show that $*$ is commutative on A.

(ii) Show that $*$ is associative on A.

(iii) Find the identity element of $*$ in A. **(2017)**

Ans. (i) (a, b) $*$ (c, d) = (ad + bc, bd)

Now, (c, d) $*$ (a, b) = (cb + da, db) = (ad + bc, bd) = (a, b) $*$ (c, d)

$\Rightarrow$ $*$ is commutative.

(ii) [(a, b) $*$ (c, d)] $*$ (e, f) = (ad + bc, bd) $*$ (e, f) = (adf + bcf + bde, bdf)

(a,b) $*$ [(c, d) $*$ (e, f)] = (a, b) $*$ (cf + de, df) = (adf + bcf + bde, bdf)

$\Rightarrow$ $*$ is associative.

Let (e_1, e_2) be the identity element of A.

$\Rightarrow (a, b) * (e_1, e_2) = (a, b) = (e_1, e_2) * (a, b)$

$\Rightarrow (ae_2 + be_1, be_2) = (a, b) = (e_1b + e_2a, e_2b)$

$\Rightarrow ae_2 + be_1 = a$ and $be_2 = b \Rightarrow e_1 = 0, e_2 = 1$

$\Rightarrow$ (0, 1) is the identity on A.

44. Let $f : N \rightarrow N$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \rightarrow S$ is invertible (where S is range of f). Find the inverse of f and hence find $f^{-1}(31)$ and f^{-1} (87). **(2016, 2015)**

Ans. Let $x_1, x_2 \in N$ and $f(x_1) = f(x_2)$

$4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)(4x_1 + 4x_2 + 12) = 0$

$\Rightarrow x_1 - x_2 = 0$ or $x_1 = x_2$ as $4x_1 + 4x_2 + 12 \neq 0,\ x_1, x_2 \in N$

$\therefore f$ is a 1 – 1 function

$f : N \rightarrow S$ is onto as co-domain = range

Hence f is invertible. $y = 4x^2 + 12x + 15 = (2x + 3)^2 + 6$

$$\Rightarrow x = \frac{\sqrt{y-6}-3}{2}$$

$$\therefore f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}, \quad y \in s$$

$$f^{-1}(31) = \frac{\sqrt{31-6}-3}{2} = 1$$

$$f^{-1}(87) = \frac{\sqrt{87-6}-3}{2} = 3$$

45. Let A = R - {3} and B = R - {1}. Consider the function f : A $\rightarrow$ B defined by $f(x) = \frac{x-2}{x-3}$. Prove that f is one - one and on to function. Hence find f^{1} (x). **(2016)**

Ans. Let $x_1, x_2 \in$ R – {3} such that $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow x_1x_2 - 2x_2 + \not{6} - 3x_1 = x_1x_2 - 2x_1 + 3x^2 + \not{6}$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$: A $\rightarrow$ B is one-one

Let y $\in$ R – {1} such that $f(x)$ = y

$$\frac{x-2}{x-3} = y$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in A, x \neq 3$$

Corresponding to every y $\in$ B, there exists $\frac{2-3y}{1-4} \in$ A, so that $\frac{2-3y}{1-y} = x \Rightarrow f$ is onto

$\therefore f^{-1}(x) = \frac{2-3x}{1-x}$

46. Let A = R x R and $*$ be the binary operation on A defined by (a, b) $*$ (c, d) = (a + c, b +d). Prove that $*$ is commutative and associative. Find the identity element for $*$ on A and hence find the inverse of elements of A. **(2016)**

Ans. Commutativity: let (a, b), (c, d) $\in$ A, then

(a, b) $*$ (c, d) = (a + c, b + d) and (c, d) $*$ (a, b) = (c + a, d + b)

for all a, b, c, d $\in$ R, a + c = c + a and b + d = d + b

$\therefore$ (a, b) $*$ (c, d) = (c, d) $*$ (a, b)

$\therefore$ $*$ is commutative on A

Associativity: For any (a, b), (c, d), (e, f) $\in$ A

{(a, b) $*$ (c, d)} $*$ (e, f) = (a + c, b + d) $*$ (e, f)

= (a + c + e, b + d + f)

Similarly (a, b) $*$ {(c, d) $*$ (e, f)} = (a + c + e, b + d + f)

$*$ is associative on A

Let (x, y) be identity element in A, then $(a, b) * (x, y) = (a, b)$

or $(a + x, b + y) = (a, b) \Rightarrow x = 9y = 0 \in R$

$\therefore$ (0, 0) is the identity element in A

Let (l, m) be inverse of (a, b) in A $\Rightarrow (a, b) * (l, m) = (0, 0)$

$\Rightarrow (a + l, b + m) = (0, 0)$

$\Rightarrow l = -a, m = -m$; which lies in R

$\therefore$ inverse is (–a, –b)

47. Consider $f : R_+ \to [-9, \propto]$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{54 + 5y} - 3}{5}\right)$. **(2015)**

Ans. $f : R_+ \to [-9, \propto]$;

$f(x) = 5x^2 + 6x - 9$;

$f^{-1}(y) = \frac{\sqrt{54 + 5y} - 3}{5}$

$f \text{ o } f^{-1}(y) = \left\{\frac{\sqrt{54 + 5y} - 3}{5}\right\}^2 + 6\left\{\frac{\sqrt{54 + 5y} - 3}{5}\right\} - 9 = y$

$f^{-1} \text{ o } f(x) = \frac{\sqrt{54 + 5(5x^2 + 6x - 9) - 6} - 3}{5} = x$

Hence 'f' is invertible with $f^{-1}(y) = \frac{\sqrt{54 + 5y} - 3}{5}$

48. A binary operation $*$ is defined on the set $x = R - \{-1\}$ by

$x * y = x + y + xy, \forall x, y \in X$.

Check whether $*$ is commutative and associative. Find its identity element and also find the inverse of each element of X. **(2015)**

Ans. (i) commutative : let $x, y \in R - \{-1\}$ then

$x * y = x + y + xy = y + x + yx = y * x \therefore x$ is commutative

(ii) Associative : let $x, y, z \in R - \{-1\}$ then

$x * (y * z) = x * (y + z + yz) = x + (y + z + yz) + x(y + z + yz)$

$= x + y + z + xy + yz + zx + xyz$

$(x * y) * z = (x + y + xy) * z = (x + y + xy) + z + (x + y + xy) . z$

$= x + y + z + xy + yz + zx + xyz$

$x * (y * z) = (x * y) * z$

$\therefore$ * is Associative

(iii) Identity Element : let $e \in R - \{-1\}$ such that $a * e = e * a = a \ \forall \ a \in R - \{-1\}$

$\therefore a + e + ae = a \Rightarrow e = 0$

(iv) Inverse : let $a * b = b * a = e = 0$; $a, b \in R - \{-1\}$

$\Rightarrow a + b + ab = 0$

$\therefore b = \frac{-a}{1+a}$ or $a^{-1} = \frac{-a}{1+a}$

49. If the function $f : R \to R$ be defined by $f(x) = 2x - 3$ and $g : R \to R$ by $g(x) = x^3 + 5$, then find the value of $(fog)^{-1}(x)$. **(2015)**

Ans. Let y = (fog) (x) [say y = h (x)]

$$f[g(x)] = f(x^3 + 5)$$

$$= 2(x^3 + 5) - 3 = 2x^3 + 7$$

$$\therefore x = \sqrt[3]{\frac{y-7}{2}} = h^{-1}(y)$$

$$\therefore (fog)^{-1} = \sqrt[3]{\frac{x-7}{2}}$$

50. Let A = Q × Q, where Q is the set of all rational numbers, and * be a binary operation defined on A by (a, b) * (c, d) = (ac, b + ad), for all (a, b) (c, d) $\in$ A. **(2015)**

Find (i) the identity element in A. (ii) the invertible element of A.

Ans. Let (x, y) be the identity element in Q × Q, then

$$(a, b) * (x, y) = (a, b) = (x, y) * (a, b) \ \forall \ (a, b) \in Q \times Q$$

$$\Rightarrow (ax, b + ay) = (a, b)$$

$$\Rightarrow a = ax \text{ and } b = b + ay$$

$$\Rightarrow x = 1 \text{ and } y = 0$$

$\therefore$(1, 0) is the identity element in Q x Q

Let (a, b) be the invertible element in Q × Q, then

there exists $(\alpha, \beta) \in$ Q x Q such that

$$(a, b) * (\alpha, \beta) = (\alpha, \beta) * (a, b) = (1, 0)$$

$$(a\alpha, b + a\beta) = (1, 0)$$

$$\Rightarrow \alpha = \frac{1}{a}, \beta = \frac{-b}{a}$$

the invertible element in A is $\left(\frac{1}{a}, -\frac{b}{a}\right)$

51. Check whether the operation * defined on the set A = R × R as **(2015)**

(a, b) * (c, d) = (a + c, b + d)

is a binary operation or not, where R is the set of all real numbers. If it is a binary operation, is it commutative and associative too ? Also find the identity element of *.

Ans. $(a, b) * (c, d) = (a + c, b + d) \ \forall\ a, b, c, d \in R$

Since $a + c \in R$ and $b + d \in R \Rightarrow (a + c, b + d) \in R \times R$

i.e. '$*$' is binary operation

For commutative

$$\text{consider } (c, d) * (a, b) = (c + a, d + b)$$
$$= (a + c, b + d)$$
$$= (a, b) * (c, d)$$

$\Rightarrow$ '$*$' is commutative

For Associative

$$\text{Let } (a, b), (c, d), (e, f) \in R \times R = A$$
$$[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f)$$
$$= (a + c + e, b + d + f) \quad \rightarrow(1)$$
$$\text{again } (a, b) * [(c, d)] * (e, f)] = (a, b) * (c + e, d + f)$$
$$= (a + c + e, b + d + f) \quad \rightarrow(2)$$

(1) & (2) $\Rightarrow$ '$*$' is associative

For identity element

Let $(e_1, e_2) \in R \times R$ be the identity element (if exists)

$$\text{then } (a, b) * (e_1, e_2) = (a, b) = (e_1, e_2) * (a, b)$$

$(e_1, e_2) = (0, 0) \in R \times R$

52. Let A = { –1, 0, 1, 2}, B = { – 4, – 2, 0, 2} and f, g : A → B be functions defined by $f(x) = x^2 - x, x \in A$ and $g(x) = 2\ |\ x - \frac{1}{2}\ | - 1, x \in A$. Find gof (x) and hence show that f = g = gof. **(2015)**

Ans. $f(x) = x^2 - x\ ; x \in \{-1, 0, 1, 2\}$

$f(-1) = 2, f(0) = 0, f(1) = 0, f(2) = 2$

$$\therefore\ f = \{(-1, 2), (0, 0), (1, 0), (2, 2)\}$$

$$g(x) = 2\left|x - \frac{1}{2}\right| - 1 \ \forall\ x \in \{-1, 0, 1, 2\}$$

g (– 1) = 2, g (0) = 0, g (1) = 0, g (2) = 2

$$g = \{(-1, 2), (0, 0), (1, 0), (2, 2)\}$$

$$(g\,of)\,(x) = g\,(f(-1), g\,(f(0), g\,(f(1), g\,(f(2) \;\forall\, x \in A$$

$$= 2, 0, 0, 2$$

$$g\,of = \{(-1, 2), (0, 0), (1, 0), (2, 2)\}$$

$$\text{Hence } f = g = g\,o\,f$$

ENTRANCE CORNER

1. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of A x B having 3 or more elements is

   (a) 211 (b) 256 (c) 220 (d) 219 **(2013)**

2. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is

   (a) $(-\infty, 0)$ (b) $(-\infty, \infty) - \{0\}$ (c) $(-\infty, \infty)$ (d) $(0, \infty)$ **(2011)**

3. Let S = {1,2,3 ,4 }.The total number of unorderd pairs of disjoint subsets of S is equal to

   (a) 25 (b) 34 (c) 42 (d) 40 **(2010)**

4. Let A = { x,y,z}. B = {a,b,c,d } Which one of the following is not a relation from A to B

   (a) { (x,a) , (x,c) } (b) { (y, c) , (y,d) }

   (c) { (z,a) , (z,d) } (d) { (z,b) , (y,b) (a,d) }

   (e) {(x,c)} **(2010)**

5. If A = {1,2,3} , B = {1, 4 , 6 ,9 } and R is a relation from A to B defined by 'x' is greater than y' .The ange of R is

   (a) {1,4,6,9 } (b) { 4,6,9 } (c) {1} (d) None of these **(2009)**

6. Let A = {p , q ,r }.Which of the following is an equivalence relation on A

   (a) R_1= { (p,q) , (q,r) ,(p,r) (p,p)} (b) R_2= {(r,q) , (r,p) , (r,r) ,(q,q) }

   (c) R_3= { (p,p) , (q,q,),(r,r) ,(p,q) } (d) None of these **(2009)**

7. If A = { x: x is a multiple of 4 } and B = { x : x is a multiple of 6 } , then $A \cap B$ consists of all multiples of

   (a) 16 (b) 12 (c) 8 (d) 4 **(2005)**

ANSWERS

1. (d)	2. (a)	3. (d)	4. (e)
5. (c)	6. (d)	7. (b)	

✧✧✧✧✧